NONLINEAR PROBLEMS OF THE THERMAL CONDUCTIVITY EQUATION

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The asymptotic behavior of solutions of parabolic equations at infinite times has been investigated for various cases [1-6]. Two initial boundary-value problems are considered in this paper. The solution of the thermal conductivity equation with a nonlinear right-hand side is found, including also nonlinear boundary conditions. It is shown that the solution of the corresponding problem tends either to a stable, steady-state solution, or to a periodic solution, depending on the initial values of the functions and constants appearing in the conditions of the problem. Other papers [7, 8] are devoted to finding the periodic solutions of these two problems encountered in hydrodynamics (diffusion, underground hydrodynamics), and to studying the asymptotic behavior of the corresponding initial boundary problems.

1. Consider the initial boundary-value problem in the interval $0 < x < l$

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + F(u)$$
$$F(u) = \begin{cases} c & \text{for } u(x^\circ, t) < u_* \\ & (u_{**} < u_*, \quad c > 0, \ d > 0) \\ -d & \text{for } u(x^\circ, t) > u_{**} \end{cases} \tag{1.1}$$
$$u(0, t) = 0, \quad u(l, t) = 0, \quad u(x, 0) = \varphi(x)$$

Here, x° is some internal point of the interval $0 < x < l$.

It is easily seen that the equation and boundary conditions (1.1) are satisfied by the steady-state solution

$$v(x) = \frac{c}{2a^2} x(l - x) \tag{1.2}$$

$$w(x) = -\frac{d}{2a^2} x(l - x) \tag{1.3}$$

An investigation by the small perturbation method reveals that these steady-state solutions are unstable.

Let us assume for definiteness that for $\varphi(x^\circ) < u_*$ $F(u) = c$, and that for $\varphi(x^\circ) > u_*$ $F(u) = -d$. Up to some time the solution of problem (1.1) is described by the expressions

$$u_1(x, t) = \sum_{n=1}^{\infty} \{C_n \exp(-\lambda_n^2 t) - c\alpha_n [1 - \exp(-\lambda_n^2 t)]\} \sin\frac{\pi n x}{l} \tag{1.4}$$

$$u_2(x, t) = \sum_{n=1}^{\infty} \{D_n \exp(-\lambda_n^2 t) + d\alpha_n [1 - \exp(-\lambda_n^2 t)]\} \sin\frac{\pi n x}{l} \tag{1.5}$$

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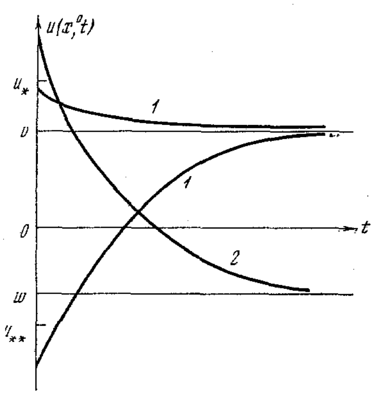

Fig. 1

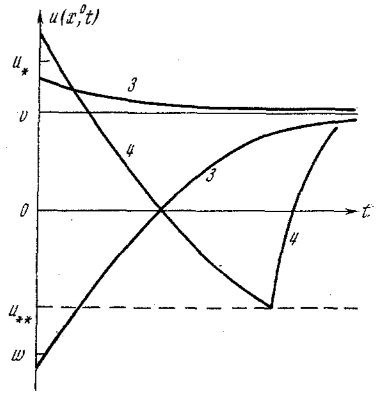

Fig. 2

Equation (1.4) holds for $\varphi(x^\circ) < u_*$, and (1.5) for $\varphi(x^\circ) > u_*$.

Here, as in [1], we introduce the notation

$$\lambda_n = \frac{\pi a n}{l}, \quad \alpha_n = \frac{2l^2[(-1)^n - 1]}{a^2\pi^3 n^3} \qquad (n = 1, 2, \ldots) \tag{1.6}$$

where C_n and D_n are Fourier coefficients of the function $\varphi(x)$, assumed to satisfy the Dirichlet condition

$$C_n = D_n = \frac{2}{l}\int_0^l \varphi(x) \sin\frac{\pi n x}{l}\, dx \tag{1.7}$$

It is easy to verify that for time t tending to infinity the solution $u_1(x, t)$, determined by Eq. (1.4), tends to v(x) and the solution $u_2(x, t)$ (1.5) to w(x). Thus, if $\varphi(x^\circ) < v(x^\circ) < u_*$, then for infinite time t the solution (1.4) tends to the steady-state solution v(x). Similarly, if $\varphi(x^\circ) > u_*$ and $w(x^\circ) > u_{**}$, (1.5) tends to the steady-state solution w(x). It can be seen that one of four situations occurs concerning the behavior of the solution of problem (1.1), depending on the reflections between the quantities u_*, u_{**}, $v(x^\circ)$, and $w(x^\circ)$.

Examples of the behavior of $u(x^\circ, t)$ for these cases are represented in Figs. 1-4, respectively.

1) $u_{**} < w(x^\circ) < v(x^\circ) < u_*$.

If $\varphi(x^\circ) < u_*$, the solution is described by Eq. (1.4). For $t \to \infty$, $u_1(x, t) \to v(x)$, i.e., the solution tends to the steady-state solution (1.2) (Fig. 1, curve 1) (in the viscous case, if $\varphi(x^\circ) < v(x^\circ)$).

If $\varphi(x^\circ) > u_*$, the solution is described by Eq. (1.5). For $t \to \infty$, $u_2(x, t) \to w(x)$, i.e., the solution tends to the steady-state solution (1.3) (Fig. 1, curve 2).

2) $w(x^\circ) < u_{**} < v(x^\circ) < u_*$.

For $\varphi(x^\circ) < u_*$, the solution is described by Eq. (1.4). For $t \to \infty$, $u_1(x, t) \to v(x)$ (Fig. 2, curve 3).

If $\varphi(x^\circ) > u_*$, the solution is described by Eq. (1.5) $[u_2(x, t)]$ up to time $t = T_1$, when $u_2(x^\circ, T_1) = u_{**}$. Starting at time $t = T_1$, the solution is described by Eq. (1.4) $[u_1(x, t)]$, in which t must be replaced by $t-T_1$, and C_n are the Fourier coefficients of the function $u_2(x, T_1)$, where $u_2(x, t)$ is given by (1.5). For $t \to \infty$, $u_1(x, t) \to v(x)$ (Fig. 2, curve 4).

3) $u_{**} < w(x^\circ) < u_* < v(x^\circ)$.

If $\varphi(x^\circ) < u_*$, the solution is described by Eq. (1.4) $[u_1(x, t)]$ up to time $t = T_1$, when $u_2(x^\circ, T_1) = u_{**}$. Starting at T_1, the solution is described by (1.5) $[u_2(x, t)]$, where t must be replaced by $t - T_1$, and D_n are the Fourier coefficients of the function $u_1(x, T_1)$. For $t \to \infty$, $u_2(x, t) \to w(x)$ (Fig. 3, curve 5).

If $\varphi(x^\circ) > u_*$, the solution is described by Eq. (1.5) $[u_2(x, t)]$. For $t \to \infty$, $u_2(x, t) \to w(x)$ (Fig. 3, curve 6).

Two more situations, similar to those described above, are still possible.

4) $w(x^\circ) < u_{**} < u_* < v(x^\circ)$.

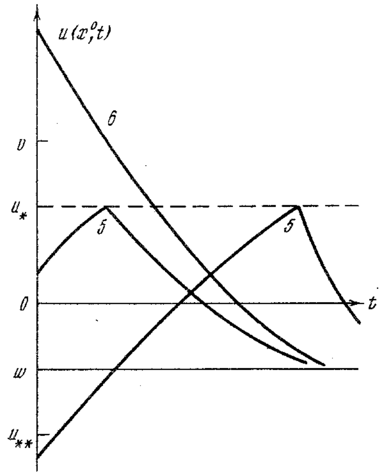

Fig. 3

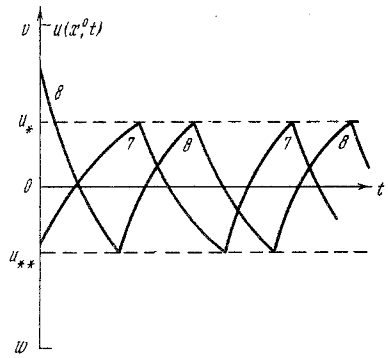

Fig. 4

If $\varphi(x^\circ) < u_*$, the solution is described by Eq. (1.4) until $t = T_1$, when $u_1(x^\circ, T_1) = u_*$, then, by Eq. (1.5) until $t = T_2$, when $u_2(x^\circ, T_2) = u_{**}$, and so forth. Consequently, the solution has an oscillatory nature. Under some restrictions it was shown [7] that for $w(x^\circ) < u_{**} < u_* < v(x^\circ)$ this solution tends to a periodic solution (Fig. 4, curve 7).

For $\varphi(x^\circ) > u_*$, the solution has again an oscillatory nature, described by Eq. (1.5), then, by Eq. (1.4), and becoming periodic for infinite time (Fig. 4, curve 8).

Thus, for infinite t the solution of the initial boundary-value problem tends either to one of the two stable, steady-state solution or the periodic solution found earlier [7]. Consequently, a stable region occurs where self-oscillations are excited.

Periodic solutions of Rayleigh systems with various parameter distributions were considered in [9, 10].

2. Consider now another initial boundary-value problem in the interval $-l < x < 0$

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u(-l, t)}{\partial x} = F[u(-l, t)], \quad u(0, t) = 0, \quad u(x, 0) = \varphi(x) \tag{2.1}$$

Here, F(u) is a three-valued S-shaped function

$$F(u) = \begin{cases} hu + q_1 & \text{for } u < u_* \\ f(u) & \text{for } u_{**} < u < u_* \\ & (u_* > u_{**},\ q_2 > q_1,\ h \geqslant 0) \\ hu + q_2 & \text{for } u > u_{**} \end{cases} \tag{2.2}$$

where $f(u)$ is a continuous function with a negative derivative $f'(u) < 0$, so that $f(0) = 0$, $f(u_*) = hu_* + q_1$, $f(u_{**}) = hu_{**} + q_2$. (Here, and in what follows, the prime denotes differentiation.)

The problem (2.1), (2.2) has the following steady-state solution:

$$u \equiv 0, \quad u = A_i x \quad (u_{**} \leqslant -A_i l \leqslant u_*, \quad i = 1, 2, \ldots) \tag{2.3}$$

$$v(x) = q_1 x / (1 + hl) \tag{2.4}$$

$$w(x) = q_2 x / (1 + hl) \tag{2.5}$$

A_i is the root of the equation

$$A_i = f(-A_i l)$$

It is easy to see that the steady-state solutions (2.4) and (2.5) are stable.

The steady-state solution (2.3) corresponds to the branch $F(u) = f(u)$. As in the case $u \equiv 0$ [11], the solution $u = A_i x$ is stable, if the inequality $-lf'(-A_i l) < 1$ is satisfied, and is unstable, if $-lf'(-A_i l) > 1$. Due to (2.1), (2.2) the study of instability of the steady-state solution $u = A_i x$ is reduced to looking for a perturbation $u' = (x, t)$ of the form $e^{\lambda_n t}\psi_n(x)$ and to solving the equation

$$a^2 \psi_n''(x) = \lambda_n \psi_n(x) \tag{2.6}$$

with boundary conditions

$$\psi_n'(-l) = f'(-A_i l)\psi_n(-l), \quad \psi_n(0) = 0 \tag{2.7}$$

so as to determine the function $\psi_n(x)$. Equation (2.6) and the second of conditions (2.7) are satisfied by the functions $\psi_n(x) = \sin \omega_n x$ and $\psi(x) = \operatorname{sh} \Omega x$. In the first case, we obtain from (2.6) and (2.7), the following equations for the quantities ω_n and λ_n:

$$\operatorname{tg} \omega_n l = -\frac{\omega_n}{f'(-A_i l)}, \quad \lambda_n = -a^2\omega_n^2 \quad (n = 1, 2, \ldots) \tag{2.8}$$

and, in the second case, we have

$$\operatorname{th} \Omega l = -\frac{\Omega}{f'(-A_i l)}, \quad \lambda = a^2\Omega^2 \tag{2.9}$$

Thus, in the first case the inequality $\lambda_n < 0$ (stable solution) is satisfied, and in the second case, the inequality $\lambda > 0$ (unstable solution). It is clear from the first equality of (2.9) that instability occurs for

$$-lf'(-A_i l) > 1 \tag{2.10}$$

In particular, for the steady-state solution $u \equiv 0$, which holds for any function $f(u)$ with the properties described above. The stability and instability conditions acquire the form

$$-lf'(0) < 1, \quad -lf'(0) > 1 \tag{2.11}$$

We turn now to the case when s, the number of roots of the equation $A = f(-Al)$, is finite, while $-lf'(-A_i l) \neq 1$ ($i = 1, 2, \ldots, s$). For definiteness let the following inequalities be satisfied:

$$u_{**} < -A_1 l < -A_2 l < \ldots < -A_{s-1} l < -A_s l < u_*$$

Stable steady-state solutions will then alternate with unstable ones. Let the steady-state solution $u = A_i x$ be unstable. In that case, $u = A_{i-1}x$ and $u = A_{i+1}x$ are stable. As was done for the case $u \equiv 0$ [11], the nonlinear integral equation to which the initial boundary-value problem (2.1), (2.2) is reduced, with the replacement $F(u) = f(u)$, can be compared with the corresponding linear integral equation, for which $F(u) = A_i - l^{-1}(u + A_i l)$.

It appears that if $-A_{i-1} l < \varphi(-l) < -A_i l$, then, for $t \to \infty$, the solution of the nonlinear problem tends to $A_{i-1}x$. If $-A_i l < \varphi(-l) < -A_{i+1} l$, the solution tends to $A_{i+1}x$. If the steady-state solution $u = A_s x$, and if $-A_s l < \varphi(-l) < u_*$, the solution of the problem tends to $A_s x$. Similarly, if $u = A_1 x$, and if $u_{**} < \varphi(-l) < -A_1 l$, the solution tends to $A_1 x$. If the steady-state solution $A_s x$ is unstable, and if $-A_s l < \varphi(-l) < u_*$, the function $u(-l, t)$ reaches the value u_* at a finite interval T_1. Similarly, if the steady-state solution $u = A_1 x$ is unstable, then, for $u_{**} < \varphi(-l) < -A_1 l$, the function $u(-l, t)$ reaches the value u_{**} at time T_1. Starting from this moment, the function $F(u)$ appearing in the boundary condition acquires either the value $hu + q_1$, or $hu + q_2$. The branch $F(u) = f(u)$ should not be considered now. The solution of the problem (2.1), (2.2) is described by the expressions

$$u_1(x, t) = \frac{q_1 x}{1 + hl} + \sum_{n=1}^{\infty} C_n \exp(-\lambda_n^2 t) \sin \alpha_n x \tag{2.12}$$

$$u_2(x, t) = \frac{q_2 x}{1 + hl} + \sum_{n=1}^{\infty} D_n \exp(-\lambda_n^2 t) \sin \alpha_n x \tag{2.13}$$

where λ_n are the roots of the equation

$$\operatorname{tg} \frac{\lambda_n l}{a} = -\frac{\lambda_n}{ah} \quad \left(\alpha_n = \frac{\lambda_n}{a}, \; n = 1, 2, \ldots\right) \tag{2.14}$$

C_n and D_n are the Fourier coefficients of the functions $\varphi(x) - q_1 x/(1 + hl)$ and $\varphi(x) - q_2 x/(1 + hl)$, respectively, [8]. In Eqs. (2.12), (2.13), the initial moment of time $t = 0$ should be taken as $t = T_1$, and the initial function $\varphi(x)$ as the function $u(x, T_1)$. Here, $u(x, t)$ is the solution of the problem (2.1), (2.2) for the branch $F(u) = f(u)$. If $-A_s l < \varphi(-l) < u_*$, Eq. (2.13) is used, and if $u_{**} < \varphi(-l) < -A_1 l$, Eq. (2.12) is used.

If the branches $F(u) = hu + q_j$ ($j = 1, 2$) are considered as functions of $F(u)$, we assume again for definiteness that for $\varphi(-l) < u_*$, the solution (2.12) holds, and for $\varphi(-l) > u_*$ solution (2.13) holds.

As in the first problem, if $\varphi(-l) < v(-l) < u_*$, then for $t \to \infty$, $u_1(x, t) \to v(x)$; if $\varphi(-l) > u_*$ and $w(-l) > u_{**}$, then for $t \to \infty$, $u_2(x, t) \to w(x)$. Thus, putting $x^\circ = -l$, for $t \to \infty$, we note that the solution of problem (2.1), (2.2) either tends to one of the stable, steady-state solutions (2.3), or all possible shapes of the solution of problem (1.1), described in the previous sections (Figs. 1-4), are conserved for this problem. The oscillatory solution earlier obtained [8] is represented in Fig. 4, and it is seen that under some conditions it tends to a periodic solution.

The solution of the second problem thus also tends either to one of the steady-state solutions (2.3)-(2.5), or to a periodic solution.

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